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A commentary on the 2x2 matrix KdV equation

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Abstract

This work demonstrates that the cnoidal wave solution of the 2x2 matrix KdV equation turns out to be a logarithmic derivative of a 2x2-matrix, whereby the matrix elements are Neville theta functions.

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Let U and V denote linear operators, and U+V and UV are the sum and product, respectively. We call

$$U_t + 3U_xU_x + U_{xxx} = A \tag{1}$$

the operator valued KdV equation with unknown operator U and given constant operator A. The classical work on operator valued soliton-like equations is Marčenko [4]. For further reading please see [1] and [6].

Let $\theta_p\,(z|\tau)$, $p = s, c, d, n$ denote the four Neville theta functions, see [3, 20.1]. Here we assume that the argument z is real and $\Im\,\tau > 0$ applies to the lattice parameter τ. Derivatives regarding z are abbreviated with ${\theta_p}'$. We will require the properties

$$\theta_s^2 + \theta_c^2 = \theta_n^2\,, \tag{2}$$

see [3, 20.4.6, 20.7.3] and

$${\theta_s}'\,\theta_c - \theta_s\,{\theta_c}' = \theta_d\theta_n\,, \tag{3}$$

see [3, 20.11] and [2, 3.1.]. We state

$$\Theta\,(z|\tau) = \begin{bmatrix} \theta_c\,(z|\tau) & -\theta_s\,(z|\tau) \\ \theta_s\,(z|\tau) & \theta_c\,(z|\tau) \end{bmatrix}\,. \tag{4}$$

Then

$$\Theta^{-1}\Theta' = \begin{bmatrix} {\theta_n}'/\theta_n & -\theta_d/\theta_n \\ \theta_d/\theta_n & {\theta_n}'/\theta_n \end{bmatrix}, \tag{5}$$

because

$$\Theta^{-1}\Theta' = \frac{1}{\theta_n^2}\begin{bmatrix} \theta_c & -\theta_s \\ \theta_s & \theta_c \end{bmatrix}\begin{bmatrix} {\theta_c}' & -{\theta_s}' \\ {\theta_s}' & {\theta_c}' \end{bmatrix} = \frac{1}{\theta_n^2}\begin{bmatrix} \theta_c\,{\theta_c}' + \theta_s\,{\theta_s}' & -{\theta_s}'\,\theta_c + \theta_s\,{\theta_c}' \\ {\theta_s}'\,\theta_c - \theta_s\,{\theta_c}' & \theta_c\,{\theta_c}' + \theta_s\,{\theta_s}' \end{bmatrix}.$$

To simplify the further considerations, we introduce the Jacobian elliptic function

$$\mathrm{dn}\,(z,k) = \frac{\theta_d\,(z,|\tau)}{\theta_n\,(z|\tau)}, \qquad \tau = i\frac{K\,(k')}{K\,(k)}, \tag{6}$$

see [3, 22.2.11, 12]. Here $K\,(k)$ and $E\,(k)$ denote the complete elliptic integrals of the first and second orders, respectively, and k and k' are complementaries moduli to each other . Let $k' = \sqrt{1-k^2}$. Then it holds that

$$\frac{d}{dz}\ln\theta_n\,(z|\tau) = \int_0^z \mathrm{dn}^2\,(\zeta,k)\,d\zeta - \frac{E\,(k)}{K\,(k)}z, \tag{7}$$

see [3, 22.16.17,30]. Hence

$$\Theta^{-1}\Theta' = \begin{bmatrix} \int_0^z \mathrm{dn}^2\,(\zeta,k)\,d\zeta & \text{-dn}\,(z,k) \\ \mathrm{dn}\,(z,k) & \int_0^z \mathrm{dn}^2\,(\zeta,k)\,d\zeta \end{bmatrix} - \frac{E\,(k)}{K\,(k)}z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

Note that, we have actually eliminated the θ symbols from the logarithmic derivative of Θ. Thus we can use the well-known differential equations

$$\left(\mathrm{dn}'\right)^2 = \left(1-\mathrm{dn}^2\right)\left(\mathrm{dn}^2 - k'^2\right) \tag{9}$$

and

$$\mathrm{dn}'' = \left(1+k'^2\right)\mathrm{dn} - 2\mathrm{dn}^3. \tag{10}$$

see [3, 22.13.3, 15]. From here it follows

$$\left(\mathrm{dn}^2\right)'' = -6\mathrm{dn}^4 + 4\left(1+k'^2\right)\mathrm{dn}^2 - 2k'^2. \tag{11}$$

From now on, we set

$$c = 6\left(1+\frac{E\,(k)}{K\,(k)}\right) - \left(1+k'^2\right) \tag{12}$$

and

$$A = \left\{k'^2 + 3\left(1+\frac{E\,(k)}{K\,(k)}\right)^2\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

The result then follows from the lemma.

Lemma 1. *The logarithmic derivative* $U(z|\tau) = \Theta^{-1}\frac{d}{dz}\Theta(z|\tau)$ *of*

$$\Theta(z|\tau) = \begin{bmatrix} \theta_c(z|\tau) & -\theta_s(z|\tau) \\ \theta_s(z|\tau) & \theta_c(z|\tau) \end{bmatrix}$$

is a solution of

$$cU' + 3U'U' + U''' = A\,. \tag{14}$$

Proof. From (8), (9) we obtain

$$\begin{aligned} U'^2 &= \begin{bmatrix} \mathrm{dn}^2 - \left(1 + \frac{E(k)}{K(k)}\right) & -\mathrm{dn}' \\ \mathrm{dn}' & \mathrm{dn}^2 - \left(1 + \frac{E(k)}{K(k)}\right) \end{bmatrix}^2 \\ &= \begin{bmatrix} 2\mathrm{dn}^4 - \left\{2\left(1 + \frac{E(k)}{K(k)}\right) + \left(1 + k'^2\right)\right\}\mathrm{dn}^2 + \left\{\left(1 + \frac{E(k)}{K(k)}\right)^2 + k'^2\right\} & -\cdots \\ 2\left(\mathrm{dn}^2 - \left(1 + \frac{E(k)}{K(k)}\right)\right)\mathrm{dn}' & \cdots \end{bmatrix}, \end{aligned}$$

and from (10), (11) we have

$$U''' = \begin{bmatrix} \left(\mathrm{dn}^2\right)'' & -\mathrm{dn}''' \\ \mathrm{dn}''' & \left(\mathrm{dn}^2\right)'' \end{bmatrix} = \begin{bmatrix} -6\mathrm{dn}^4 + 4\left(1 + k'^2\right)\mathrm{dn}^2 - 2k'^2 & -\cdots \\ \left(1 + k'^2\right)\mathrm{dn}' - 6\mathrm{dn}^2\mathrm{dn}' & \cdots \end{bmatrix}.$$

$\square$

Proposition 2. *Let* $a\epsilon\mathrm{R}$*. The logarithmic derivative* $U(x,t|\tau) = \Theta^{-1}\Theta_x\left(ax + a^3ct|\tau\right)$ *is a solution of*

$$U_t + 3U_xU_x + U_{xxx} = a^4A\,.$$

Remark 3. In bordline cases $k \to 1$ it holds $K(k) \to \infty$, $K(k') \to \frac{\pi}{2}$, and $\mathrm{dn}(z,k) \to \mathrm{sech}(z)$, see for instance [3, 22.5.ii, 4]. The identities (3)-(12) are maintained, if θ_p, $p = s, c, d, n$ replaced by sinh, 1, 1, sech, respectively. Thus, the assumptions of the proposition are satified, i.e. the logarithmic derivative $U(x,t) = \Theta^{-1}\Theta_x\left(ax + a^3ct\right)$ of

$$\Theta(z) = \begin{bmatrix} 1 & -\sinh(z) \\ \sinh(z) & 1 \end{bmatrix}$$

is a solution of (1) with $c = 5$ and $A = 3a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. As expected we obtain the well-known 1-soliton solution

$$U(x,t) = \begin{bmatrix} \tanh\left(ax + a^3ct\right) & -\mathrm{sech}\left(ax + a^3ct\right) \\ \mathrm{sech}\left(ax + a^3ct\right) & \tanh\left(ax + a^3ct\right) \end{bmatrix}$$

of (1) .

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